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Models for Mandel's formula

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Abstract Photon-counting models based on Poisson transforms was pioneered by Mandel (Proc. Phys. Soc. **74**, 233–243 1959). This note derives a collection of some sixteen flexible models for the formula suggested by Mandel (Proc. Phys. Soc. **74**, 233–243 1959). The corresponding estimation procedures are also derived by the method of moments. We expect that this work could serve as a useful reference for photon-counting problems.

Keywords Photon-counting models · Poisson transforms · Method of moments

1 Introduction

The Poisson transform intervenes in optics for the statistical study of a random field at very low light levels, in photon counting mode. In the limits of the semiclassical theory for photoelectric detection of light [1], the observed number of photons per pixel fluctuates for two reasons; one is the random nature of the pattern itself that presents variable intensities and the other is the quantum nature of the photo–detection that may produce different numbers of photo–events from an identical intensity of the incident wave.

Given a value Λ of the integrated intensity falling into a pixel and expressed as a real number in units of photo–counts, the actual number n of photons detected per pixel is a random integer variable that follows a Poisson distribution $\Pr(n \mid \Lambda)$ of the form:

$$\Pr(n \mid \Lambda) = \frac{\lambda^n \exp(-\lambda)}{n!}. \qquad (1)$$

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For the sake of simplicity of notation, we assume a detector with 100% quantum efficiency, so that the mean of n equals that of Λ.

The probability $p(n)$ of detecting n photons in a pixel, regardless of any particular value Λ, is the unconditional probability:

$$p(n) = \int_0^\infty \frac{\lambda^n \exp(-\lambda)}{n!} g(\lambda) d\lambda, \tag{2}$$

where $g(\cdot)$ is the probability density function (pdf) of Λ. This expression is known as the Poisson transform and was first derived by [2].

In this note, we derive a comprehensive collection of formulas for (2) by taking $g(\lambda)$ to belong to sixteen flexible families. For each $g(\cdot)$, we derive the corresponding $p(n)$ given by Eq. 2 as well as provide estimators of the associated parameters obtained by the method of moments. See Sect. 2. Some graphical comparison of the models for $p(n)$ and a numerical test for comparison are described in Sects. 3 and 4. The details of the analytical calculations are not given here and can be obtained from the author.

The calculations of this note use several special functions, including the integral cosine defined by

$$\mathrm{ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt,$$

the integral sine defined by

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t} dt,$$

the incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t)\, dt,$$

the error function defined by

$$\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt,$$

the modified Bessel function of the third kind defined by

$$K_\nu(x) = \frac{x^\nu \Gamma(1/2)}{2^\nu \Gamma(\nu + 1/2)} \int_1^\infty \exp(-xt)(t^2 - 1)^{\nu - 1/2} dt,$$

the parabolic cylinder function defined by

$$D_p(x) = \frac{\exp(-x^2/4)}{\Gamma(-p)} \int_0^\infty \exp\{-(tx + t^2/2)\} t^{-(p+1)} dt,$$

the ${}_1F_1$ hypergeometric function (also known as the confluent hypergeometric function) defined by

$$ {}_1F_1\,(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}, $$

the ${}_1F_2$ hypergeometric function defined by

$$ {}_1F_2\,(a;b,c;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k\,(c)_k} \frac{x^k}{k!}, $$

and, the Kummer function defined by

$$ \Psi\,(a,b;x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}\,{}_1F_1(a;b;x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b}\,{}_1F_1(1+a-b;2-b;x), $$

where $(f)_k = f(f+1)\ldots(f+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in [3] and [4].

The estimation by the method of moments requires the following notation: for a random sample $\lambda_1,\ldots,\lambda_n$ of λ, define

$$ \beta_1 = \frac{\{2\mu_1^3 - 3\mu_1\mu_2 + \mu_3\}^2}{\{\mu_2 - \mu_1^2\}^3} \tag{3} $$

and

$$ \beta_2 = \frac{-3\mu_1^4 + 6\mu_1^2\mu_2 - 4\mu_1\mu_3 + \mu_4}{\{\mu_2 - \mu_1^2\}^2} \tag{4} $$

as the sample skewness and sample kurtosis, respectively, where μ_j is the jth sample moment defined by $\mu_j = (1/n)\sum_{i=1}^{n} \lambda_i^j$ for $j = 1, 2, 3, 4$.

2 Photon-counting models

In this section, we provide a collection of formulas for (2) by taking $g(\cdot)$ to belong to sixteen flexible families. The estimators for the parameters of $g(\cdot)$ determined by the method of moments are also given.
Beta distribution: if g takes the form

$$ g(\lambda) = \frac{\lambda^{p-1}(1-\lambda)^{q-1}}{B(p,q)} \tag{5} $$

for $0 < \lambda < 1$ then

$$p(n) = \frac{B(p+n,q)}{n!B(p,q)}\, {}_1F_1\left(p+n;\, p+q+n;\, -1\right). \tag{6}$$

If g takes the form of the generalized beta distribution given by

$$g(\lambda) = \frac{(\lambda-a)^{p-1}(b-\lambda)^{q-1}}{B(p,q)(b-a)^{p+q-1}} \tag{7}$$

for $a < \lambda < b$ then

$$p(n) = \frac{\exp(-a)}{n!B(p,q)} \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(b-a)^k B(k+p,q)\, {}_1F_1\left(k+p;\, k+p+q;\, -(b-a)\right).$$

The moment estimators of p and q are the solutions of the equations

$$\sqrt{\beta_1}\sqrt{pq}(p+q+2) = 2(q-p)\sqrt{p+q+1}$$

and

$$\beta_2 pq(p+q+2)(p+q+3) = 6\{p^3 - p^2(2q-1) + q^2(q+1) - 2pq(q+2)\},$$

where β_1 and β_2 are the sample skewness and the sample kurtosis given by Eqs. 3 and 4, respectively. For the generalized form given by (7), the moment estimators of a, b, p and q are the solutions of the equations

$$\mu_1 = \sum_{k=0}^{1} \frac{a^k(b-a)^{1-k}B(p+1-k,q)}{B(p,q)},$$

$$\mu_2 = \sum_{k=0}^{2} \binom{2}{k} \frac{a^k(b-a)^{2-k}B(p+2-k,q)}{B(p,q)},$$

$$\mu_3 = \sum_{k=0}^{3} \frac{a^k(b-a)^{3-k}B(p+3-k,q)}{B(p,q)},$$

and

$$\mu_4 = \sum_{k=0}^{4} \frac{a^k(b-a)^{4-k}B(p+4-k,q)}{B(p,q)},$$

where μ_j, $j = 1, 2, 3, 4$ are the first four sample means.
Uniform distribution: if g takes the form

$$g(\lambda) = \frac{1}{b - a}$$

for $a < \lambda < b$ then

$$p(n) = \frac{\Gamma(n+1, a) - \Gamma(n+1, b)}{n!(b-a)}.$$

The moment estimators of a and b are $\mu_1 + \sqrt{3\mu_2}$ and $\mu_1 - \sqrt{3\mu_2}$, respectively, where μ_1 and μ_2 are the first two sample means.
Beta distribution of the second kind: if g takes the form

$$g(\lambda) = \frac{\lambda^{\alpha-1}(1+\lambda)^{-\alpha-\beta}}{B(\alpha, \beta)} \tag{8}$$

for $\lambda > 0$ then

$$p(n) = \frac{\Gamma(n+\alpha)\Psi(n+\alpha, n-\beta+1; 1)}{n!B(\alpha, \beta)}.$$

The moment estimators of β and α are the solutions of the equations

$$\begin{aligned}\sqrt{\beta_1}\{\alpha(\alpha+\beta-1)\}^{3/2} = {} & 2\alpha^3(\beta-2)^{3/2} - 3\alpha^2(\alpha+1)(\beta-1)\sqrt{\beta-2} \\ & + \alpha(\alpha+1)(\alpha+2)(\beta-1)^2\sqrt{\beta-2}(\beta-3)^{-1}\end{aligned}$$

and

$$\begin{aligned}\beta_2\alpha^2(\alpha+\beta-1)^2 = {} & -3\alpha^4(\beta-2)^2 + 6\alpha^3(\alpha+1)(\beta-1)(\beta-2) \\ & - 4\alpha^2(\alpha+1)(\alpha+2)(\beta-1)^2(\beta-2)(\beta-3)^{-1} \\ & + \alpha(\alpha+1)(\alpha+2)(\alpha+3)(\beta-1)^3(\beta-2)(\beta-3)^{-1}(\beta-4)^{-1},\end{aligned}$$

where β_1 and β_2 are the sample skewness and the sample kurtosis given by Eqs. 3 and 4, respectively. Note that (8) is related to (5) by the transformation $1/\lambda - 1$.
Exponential distribution: if g takes the form

$$g(\lambda) = \beta \exp(-\beta\lambda) \tag{9}$$

for $\lambda > 0$ then

$$p(n) = \frac{\beta}{(1+\beta)^{n+1}}. \tag{10}$$

The moment estimator of β is $1/\mu_1$, where μ_1 is the sample mean.

Gamma distribution: if g takes the form

$$g(\lambda) = \frac{\beta^{\alpha}\lambda^{\alpha-1}\exp(-\beta\lambda)}{\Gamma(\alpha)} \tag{11}$$

for $\lambda > 0$ then

$$p(n) = \frac{\beta^{\alpha}\Gamma(n+\alpha)}{\Gamma(\alpha)n!(1+\beta)^{n+\alpha}}. \tag{12}$$

The moment estimators of β and α are $4/(\mu_1\beta_1)$ and $4/\beta_1$, respectively, where μ_1 is the sample mean and β_1 is the sample skewness given by Eq. 3. Note that (9) is a particular case of (11) for $\alpha = 1$.
Rayleigh distribution: if g takes the form

$$g(\lambda) = 2\beta^2\lambda\exp\{-(\beta\lambda)^2\}$$

for $\lambda > 0$ then

$$p(n) = \frac{\beta\sqrt{\pi}(-1)^{n+1}}{n!}\frac{\partial^{n+1}}{\partial q^{n+1}}\left[\exp\left(\frac{q^2}{4\beta^2}\right)\operatorname{erfc}\left(\frac{q}{2\beta}\right)\right]\Bigg|_{q=1}.$$

The moment estimator of β is $\sqrt{\pi}/(2\mu_1)$, where μ_1 is the sample mean.
Stacy distribution ($c = 2$): if g takes the form

$$g(\lambda) = \frac{2\beta^{2\alpha}\lambda^{2\alpha-1}\exp\{-(\lambda\beta)^2\}}{\Gamma(\alpha)}$$

for $\lambda > 0$ then

$$p(n) = \frac{2^{1-\alpha-n/2}\Gamma(n+2\alpha)}{n!\beta^n\Gamma(\alpha)}\exp\left(\frac{1}{8\beta^2}\right)D_{-n-2\alpha}\left(\frac{1}{\sqrt{2}\beta}\right).$$

If 2α is an integer then the above reduces to the simpler form

$$p(n) = \frac{\sqrt{\pi}(-1)^{n+2\alpha-1}\beta^{2\alpha-1}}{n!\Gamma(\alpha)}\frac{\partial^{n+2\alpha+1}}{\partial q^{n+2\alpha+1}}\left[\exp\left(\frac{q^2}{4\beta^2}\right)\operatorname{erfc}\left(\frac{q}{2\beta}\right)\right]\Bigg|_{q=1}.$$

The moment estimators of α and β are the solutions of the equations

$$\mu_1 = \frac{\Gamma(\alpha+1/2)}{\beta\Gamma(\alpha)}$$

and

$$\mu_2 = \frac{\alpha}{\beta^2},$$

where μ_1 and μ_2 are the first two sample means.
Pareto distribution of the first kind: if g takes the form

$$g(\lambda) = ak^a \lambda^{-a-1}$$

for $\lambda > k$ then

$$p(n) = \frac{ak^a \Gamma(n-a, k)}{n!}.$$

The moment estimator of a is the root of the equation

$$\sqrt{\beta_1}\sqrt{a}(a-3) = 2(a+1)\sqrt{a-2},$$

where β_1 is the sample skewness given by Eq. 3. The moment estimator of k is $\mu_1(a-1)/a$, where μ_1 is the sample mean.
Pareto distribution of the second kind: if g takes the form

$$g(\lambda) = a\lambda^a (\lambda + c)^{-a-1}$$

for $\lambda > 0$ then

$$p(n) = \frac{a\Gamma(n+a+1)c^n}{n!} \Psi\left(n+a+1, n+1; c\right).$$

The moment estimator of a is the root of the equation

$$\sqrt{\beta_1} a^{3/2} = 2(a-2)^{3/2} - 6(a-1)\sqrt{a-2} + 6(a-1)^2\sqrt{a-2}(a-3)^{-1},$$

where β_1 is the sample skewness given by Eq. 3. The moment estimator of c is $(a-1)\mu_1$, where μ_1 is the sample mean.
Inverse Gaussian distribution: if g takes the form

$$g(\lambda) = \sqrt{\frac{\phi}{2\pi}} \exp(\phi) \lambda^{-3/2} \exp\left\{-\frac{\phi}{2}\left(\lambda + \frac{1}{\lambda}\right)\right\}$$

for $\lambda > 0$ then

$$p(n) = \frac{\sqrt{2\phi}\exp(\phi)}{\sqrt{\pi} n!} \left(\frac{\phi}{2+\phi}\right)^{(2n-1)/4} K_{n-1/2}\left(\sqrt{\phi(2+\phi)}\right).$$

The moment estimator of ϕ is $9/\beta_1^2$, where β_1 is the sample skewness given by Eq. (3).

Half Gaussian distribution: if g takes the form

$$g(\lambda) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \exp\left\{-\frac{\lambda^2}{2\sigma^2}\right\} \tag{13}$$

for $\lambda > 0$ then

$$p(n) = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial q^n} \left[\exp\left(\frac{q^2\sigma^2}{2}\right) \operatorname{erfc}\left(\frac{q\sigma}{\sqrt{2}}\right)\right]\Bigg|_{q=1}. \tag{14}$$

The moment estimator of σ is $\mu_1\sqrt{\pi/2}$, where μ_1 is the sample mean.
Half logistic distribution: if g takes the form

$$g(\lambda) = \frac{2\beta \exp(-\beta\lambda)}{\left[1 + \exp(-\beta\lambda)\right]^2}$$

for $\lambda > 0$ then

$$p(n) = 2\beta \sum_{k=0}^{\infty} \binom{-2}{k} \{1 + (k+1)\beta\}^{-n-1}.$$

The moment estimator of β is $(2\ln 2/\mu_1)^{1/3}$, where μ_1 is the sample mean.
Half Cauchy distribution: if g takes the form

$$g(\lambda) = \frac{2}{\pi\alpha} \left\{1 + \frac{\lambda^2}{\alpha^2}\right\}^{-1}$$

for $\lambda > 0$ then

$$p(n) = \frac{2\alpha}{\pi n!}\Bigg[\alpha^{n-1}\left\{\sin\left(\alpha - \frac{\pi n}{2}\right)\operatorname{ci}(\alpha) - \cos\left(\alpha - \frac{\pi n}{2}\right)\operatorname{si}(\alpha)\right\} + \sum_{k=1}^{[n/2]} (n-2k)!\left(-\alpha^2\right)^{k-1}\Bigg].$$

Since this distribution has no moments estimation by the method of moments does not apply. However, the maximum likelihood estimator of α is the root of the equation:

$$\sum_{j=1}^{n} \frac{\lambda_j}{\lambda_j + \alpha^2} = \frac{1}{2}.$$

Half t distribution: if g takes the form

$$g(\lambda) = \frac{2}{\sqrt{\nu}B(1/2, \nu/2)} \left(1 + \frac{\lambda^2}{\nu}\right)^{-(1+\nu)/2}$$

for $\lambda > 0$ then

$$\begin{aligned} p(n) = \frac{2\nu^{\nu/2}}{n!B\,(1/2, \nu/2)} \Bigg[&\Gamma(n-\nu)\ {}_1F_2\left(\frac{1+\nu}{2}; 1+\frac{\nu-n}{2}, \frac{1+\nu-n}{2}; -\frac{\nu}{4}\right) \\ &+ \frac{\nu^{(n-\nu)/2}}{2} B\left(\frac{\nu-n}{2}, \frac{n+1}{2}\right)\ {}_1F_2\left(\frac{1+n}{2}; \frac{1}{2}, 1+\frac{n-\nu}{2}; -\frac{\nu}{4}\right) \\ &- \frac{\nu^{(1+n-\nu)/2}}{2} B\left(\frac{\nu-n-1}{2}, \frac{n+2}{2}\right)\ {}_1F_2\left(\frac{2+n}{2}; \frac{3}{2}, \frac{3+n-\nu}{2}; -\frac{\nu}{4}\right)\Bigg]. \end{aligned}$$

If $(1+\nu)/2$ is an integer then the above reduces to the simpler form

$$p(n) = \frac{2^{(3-\nu)/2}\nu^{\nu/2}(-1)^{n+(\nu-1)/2}}{((\nu-1)/2)!B\,(1/2, \nu/2)\,n!} \frac{\partial^n}{\partial p^n} \left(\frac{\partial}{z\partial z}\right)^{(\nu-1)/2} g(p, z)\Bigg|_{p=1, z=\sqrt{\nu}},$$

where

$$g(p, z) = \frac{\sin(pz)\mathrm{ci}(pz) - \cos(pz)\mathrm{si}(pz)}{z}.$$

The moment estimator of ν is the root of the equation

$$\frac{\sqrt{\nu}\Gamma\left((\nu-1)/2\right)}{\sqrt{\pi}\Gamma\left(\nu/2\right)} = \mu_1,$$

where μ_1 is the sample mean.
Fréchet distribution: if g takes the form

$$g(\lambda) = \frac{\gamma}{\lambda^2} \exp\left(-\frac{\gamma}{\lambda}\right)$$

for $\lambda > 0$ then

$$p(n) = \frac{2\gamma^{(n+1)/2}}{n!} K_{n-1}\left(2\sqrt{\gamma}\right).$$

Since this distribution has no moments estimation by the method of moments does not apply. However, the maximum likelihood estimator of γ is $n\{\sum_{j=1}^{n}(1/\lambda_j)\}^{-1}$.

Pearson type VI distribution: if g takes the form

$$g(\lambda) = \frac{\Gamma(p)(b-a)^{p-q-1}(\lambda-b)^q}{\Gamma(p-q-1)\Gamma(q+1)(\lambda-a)^p}$$

for $\lambda \geq b > a > 0$ then

$$p(n) = \frac{\Gamma(p)\exp(-b)}{\Gamma(p-q-1)\Gamma(q+1)n!}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}(b-a)^k\Gamma(q+k+1) \times\Psi\left(q+k+1, q-p+k+2; b-a\right).$$

The moment estimators of a, b, p and q are the solutions of the equations

$$\mu_1 = \sum_{k=0}^{1}\frac{b^k(b-a)^{1-k}\Gamma(p-q-2+k)\Gamma(q-k))}{\Gamma(p-q-1)\Gamma(q-1)},$$

$$\mu_2 = \sum_{k=0}^{2}\binom{2}{k}\frac{b^k(b-a)^{2-k}\Gamma(p-q-3+k)\Gamma(q+1-k))}{\Gamma(p-q-1)\Gamma(q-1)},$$

$$\mu_3 = \sum_{k=0}^{3}\binom{3}{k}\frac{b^k(b-a)^{3-k}\Gamma(p-q-4+k)\Gamma(q+2-k))}{\Gamma(p-q-1)\Gamma(q-1)},$$

and

$$\mu_4 = \sum_{k=0}^{4}\binom{4}{k}\frac{b^k(b-a)^{4-k}\Gamma(p-q-5+k)\Gamma(q+3-k))}{\Gamma(p-q-1)\Gamma(q-1)},$$

where μ_j, $j = 1, 2, 3, 4$ are the first four sample means.

3 Comparison study

To provide a comparison of the derived models in Sect. 2, we have plotted three of the photon count distributions in Figs. 1–3. The three cases are:

- the photon count distribution given by (6)—shown in Fig. 1—corresponding to the beta distribution for λ given by (5). The beta distribution is a widely used model for finite range data. It is clear that larger values for p and smaller values for q correspond to greater number of counts (i.e., the proportion for $n > 0$).

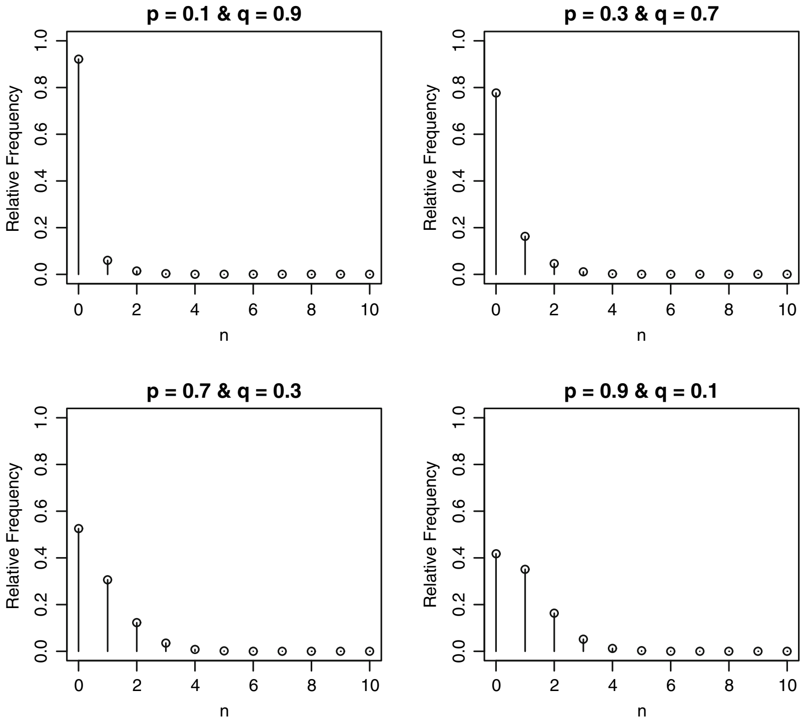

Fig. 1 Plots of the photon count distribution given by (2) for $n = 0, 1, \ldots, 10$ when $g(\lambda)$ is the beta pdf given by (5)

- the photon count distribution given by (10)—shown in Fig. 2—corresponding to the exponential distribution for λ given by (9). The exponential distribution is a widely used model for skewed data. It is clear that smaller values for β correspond to greater counts. This is intuitive because β is the reciprocal of the expected value of λ.
- the photon count distribution given by (14)—shown in Fig. 3—corresponding to the Gaussian distribution for λ given by (13). The Gaussian distribution is a widely used model for symmetric data. It is clear that larger values for σ correspond to greater counts. In fact, as $\sigma \to \infty$ the number of counts approaches the uniform distribution.

These three cases were carefully chosen to represent the sixteen models derived in Sect. 2.

4 Discussion

We have derived a collection of sixteen flexible formulas and the associated estimation procedures for photon counting models. The estimation procedures are based on the method of moments. These procedures enable the user to readily apply the models to

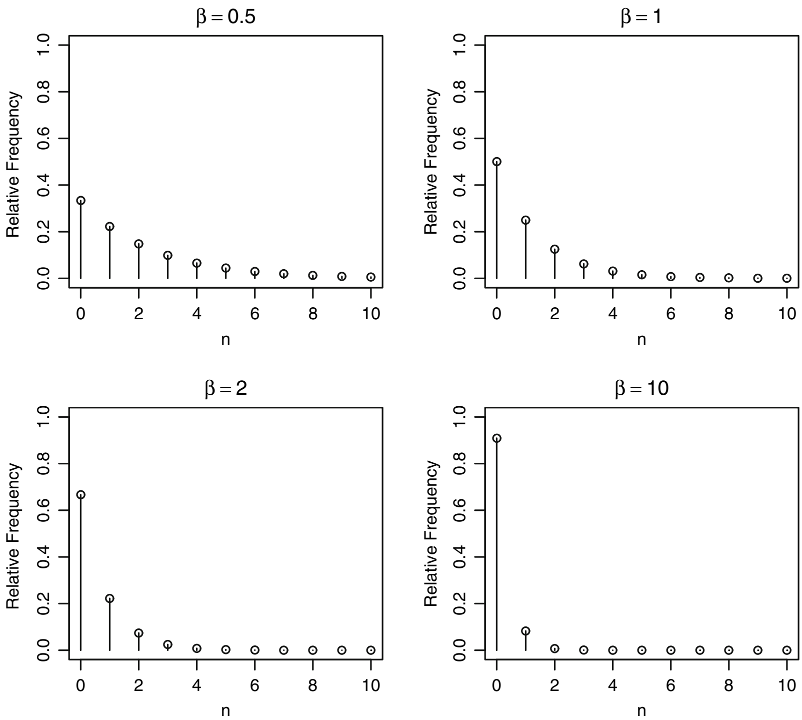

Fig. 2 Plots of the photon count distribution given by (2) for $n = 0, 1, \ldots, 10$ when $g(\lambda)$ is the exponential pdf given by (9)

the observed data. Given a data set, a practitioner could fit all of the sixteen models following the stated procedures and select the one that gives the *best* fit. There are several statistical methods for determining the *best* fit. One such method is the likelihood ratio test described in [5]. Suppose we wish to test which of the two models given by (6) and (12) is the better given some observed data. Let

$$L_1 = \prod_n p(n;\, p, q) \tag{15}$$

and

$$L_2 = \prod_n p(n;\, \alpha, \beta), \tag{16}$$

where $p(n;\, p, q)$ and $p(n;\, \alpha, \beta)$ are the photon counting models given by (6) and (12), respectively. The product in (15) and (16) is over the observed values of n. Since (6) and (12) have the same number of parameters, by the likelihood ratio test, the model based on (6) could be deemed better than that based on (12) if $L_1 > L_2$. On the other hand, (12) could be deemed better than that based on (6) if $L_2 > L_1$.

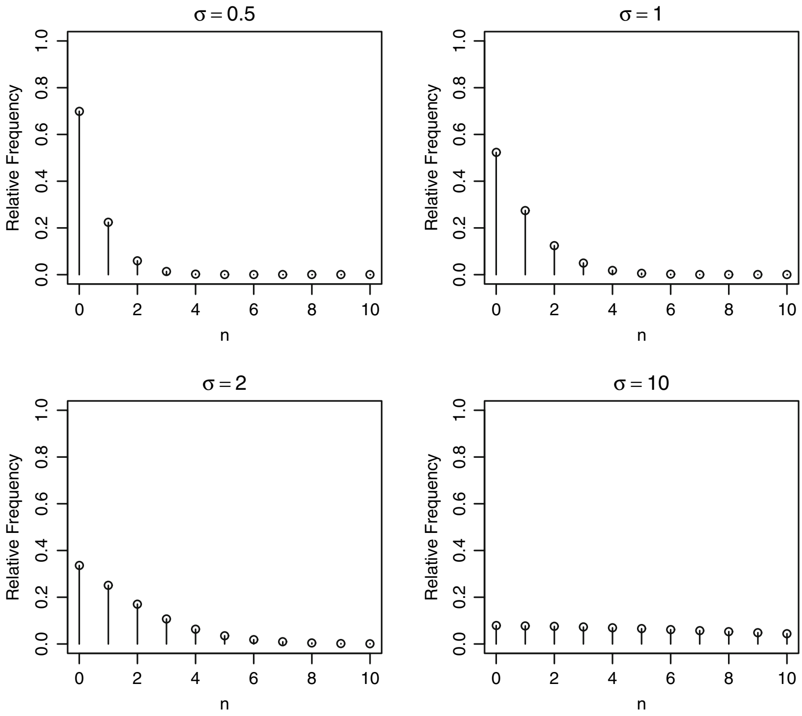

Fig. 3 Plots of the photon count distribution given by (2) for $n = 0, 1, \ldots, 10$ when $g(\lambda)$ is the half Gaussian pdf given by (13)

In conclusion, given the choice and the variety of the sixteen models, we expect that the work described could lead to improved photon-count modeling.

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